

Aspects of hidden and manifest $SL(2, \mathbb{R})$ symmetry in 2D near-horizon black-hole backgrounds.

Valter Moretti^{1,3} and Nicola Pinamonti^{2,3}

¹ Department of Mathematics, E-mail: moretti@science.unitn.it

² Department of Physics, E-mail: pinamont@science.unitn.it

³ I.N.F.N. Gruppo Collegato di Trento

University of Trento,
Faculty of Sciences,
via Sommarive 14,
I-38050 Povo (TN),
Italy.

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Abstract: The invariance under unitary representations of the conformal group $SL(2, \mathbb{R})$ of a quantum particle is rigorously investigated in two-dimensional spacetimes containing Killing horizons using de Alfaro-Fubini-Furlan's model. The limit of the near-horizon approximation is considered. If the Killing horizon is bifurcate (Schwarzschild-like near horizon limit), the conformal symmetry is hidden, i.e. it does not arise from geometrical spacetime isometries, but the whole Hilbert space turns out to be an irreducible unitary representation of $SL(2, \mathbb{R})$ and the time evolution is embodied in the unitary representation. In this case the symmetry does not depend on the mass of the particle and, if the representation is faithful, the conformal observable K shows thermal properties. If the Killing horizon is nonbifurcate (extreme Reissner-Nördstrom-like near horizon limit), the conformal symmetry is manifest, i.e. it arises from geometrical spacetime isometries. The $SL(2, \mathbb{R})$ representation which arises from the geometry selects a hidden conformal representation. Also in that case the Hilbert space is an irreducible representation of $SL(2, \mathbb{R})$ and the group conformal symmetries embodies the time evolution with respect to the local Killing time. However no thermal properties are involved, at least considering the representations induced by the geometry. The conformal observable K gives rise to Killing time evolution of the quantum state with respect to another global Killing time present in the manifold. Mathematical proofs about the developed machinery are supplied and features of the operator $H_g = -\frac{d^2}{dx^2} + \frac{g}{x^2}$, with $g = -1/4$ are discussed. It is proven that a statement, used in the recent literature, about the spectrum of self-adjoint extensions of H_g is incorrect.

1 Introduction.

Investigation about near horizon symmetries has given some hints about the structure of the quantum gravity [1, 2, 3]. In particular, it has been argued that in the near horizon limit a relevant *conformal* symmetry arises. In fact, imposing some boundary conditions, the surface deformation algebra seems to contain a Virasoro algebra. All that is based on a well known idea by Brown and Henneaux [4] who, considering the problem of the statistical nature of the black hole entropy, argued that the asymptotic anti de Sitter (*AdS*) symmetry gives rise to a central extension to the surface deformation algebra at infinity. Moreover, this framework supports the holographic nature of the gravity hypothesis [5]. Unfortunately, the attempt to get similar results in backgrounds different from *AdS* spacetime encounters some problems. It is worth stressing that the conformal symmetry turns out to be involved in quantum field theory in curved spacetime also by a different way. The celebrated *AdS/Cft* correspondence by Maldacena [6] argues that a quantum theory in d dimension of a suitable conformally invariant field describes the gravitational theory in $d + 1$ -dimensional, asymptotically *AdS*, spacetime. Generalizations to different backgrounds are under investigation [7].

In this paper we focus attention on the interplay between conformal symmetry of quantum theory and near horizon metric in two dimensional spacetimes. By *conformal invariance* we mean invariance under some $SL(2, \mathbb{R})$ (unitary) representation. We examine two relevant cases where the $SL(2, \mathbb{R})$ invariance arises. In the former case, the near horizon limit of a spacetime containing a bifurcate Killing horizon, the $SL(2, \mathbb{R})$ symmetry is “hidden”. This means that, despite such a symmetry being a natural symmetry of the physical system, it does not correspond to the background isometry group.

In the latter case, the near horizon limit of a spacetime containing a *nonbifurcate* Killing horizon, the background isometry group symmetry selects a manifest $SL(2, \mathbb{R})$ symmetry among the various hidden $SL(2, \mathbb{R})$ representations.

In both cases, the physical system is given by a spinless particle with finite mass whose wavefunction satisfies the minimally coupled Klein-Gordon equation. The hidden $SL(2, \mathbb{R})$ invariance is a straightforward consequence of the spectral decomposition of the Hamiltonian operator.

In the former case the whole Hilbert space turns out to be an irreducible $SL(2, \mathbb{R})$ unitary representation not depending on the mass of the particle. The conformal invariance singles out a class of observables which belong to the Lie algebra of the representation and are constants of motion. In the simplest case where the representation is faithful, a known conformal observable K_λ reveals a physically interesting base of proper eigenvectors. In fact, these states exhibit a thermal energy distribution and it is well known that bifurcate Killing horizons enjoy nontrivial thermodynamic properties related with Hawking’s radiation. In particular a free parameter λ can be fixed in order that the temperature associated with K_λ is Hawking-Unruh-Fulling’s one.

In the latter case we analyze quantum conformal invariance features in the bidimensional anti de Sitter spacetime AdS_2 . More precisely we confine the theory inside a region naturally delimited by a nonbifurcate Killing horizon. That spacetime is a well-known near horizon approximation of a spacetime containing an extreme Reissner-Nördstrom black hole. This background was studied in literature in relation with superconformal mechanics (e.g., see [8] and [9]). Also in

this case the Hilbert space is a irreducible representation of $SL(2, \mathbb{R})$ built up making use of the spectral representation of the Hamiltonian, but now a preferable representation is selected by the group of background isometries depending on the mass of the particle. Moreover there is no way to select a physically meaningful temperature using these manifest $SL(2, \mathbb{R})$ representations also because the faithful representation of $SL(2, \mathbb{R})$ is not allowable. On the other hand it is known by the literature that no preferable temperature for quantum field states is selected in a nonbifurcate black hole background. However, a distinguished value of the parameter λ , which determines the conformal observable K_λ , can be fixed by another way. As earlier suggested in [8] and [9], we prove, by a precise statement, that there is a choice for λ which makes K_λ the Hamiltonian generator of time evolution with respect to the global Killing time T in AdS_2 spacetime.

In the final technical section we show that the conformal invariant quantum theory of both the treated backgrounds is unitary equivalent to that studied by de Alfaro, Fubini, Furlan in [10]. In that section we give some mathematical proofs concerning the machinery used in this paper completing some statements of [10] by distinguishing between representations of $SL(2, \mathbb{R})$ and representations of its universal covering. Moreover we deal with the problem of the spectrum of the self-adjoint extensions of the differential operator $-\frac{d^2}{dx^2} - \frac{1}{4x^2}$ which recently has been discussed in the literature. We prove that the spectrum found in [11] is not correct. As a consequence, part of physical results presented in [11, 8, 12, 1, 13] could not make sense. (see the end of section 6).

2 Bifurcate Killing horizons and hidden $SL(2, \mathbb{R})$ invariance.

In this section we analyze the *hidden* $SL(2, \mathbb{R})$ invariance of a quantum theory in near the horizon approximation of a bifurcate Killing horizon black hole (e.g., a Schwarzschild black hole or a nonextremal charged black hole).

Near the horizon, i.e. $r \sim r_h > 0$, the metric takes the form

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Sigma, \quad (1)$$

where Σ denotes angular coordinates. As the horizon is bifurcate, $A'(r_h)/2 \neq 0$ and we can use the following approximation $A(r) = A'(r_h)(r - r_h) + O((r - r_h)^2)$. If $\kappa = A'(r_h)/2$ denotes the surface gravity, in the limit $r \rightarrow r_h$ the metric becomes

$$ds^2 = -\kappa^2 y^2 dt^2 + dy^2 + r^2(y) d\Sigma, \quad (2)$$

where $r = r_h + A'(r_h)y^2/2$ and $x \in [0, +\infty)$. In the following we drop the angular part $r^2(x)d\Sigma$ and consider the metric of the two-dimensional toy model given by the *Rindler spacetime*, \mathcal{M}_R^2

$$ds_R^2 = -\kappa^2 y^2 dt^2 + dy^2, \quad (3)$$

with $t \in (-\infty, +\infty)$, $y \in (0, +\infty)$. The isometry group of \mathcal{M}_R^2 is generated by three Killing fields: The Lorentz boost generator ∂_t , a generator of Minkowski time displacements ∂_T and the

generator of orthogonal space displacements ∂_X . The generated Lie algebra is not $sl(2, \mathbb{R})$ because $[\partial_T, \partial_X] = 0$ and this is not compatible with the structure constants of $sl(2, \mathbb{R})$. Therefore, if a physical system propagating in \mathcal{M}_R^2 turns out to be invariant under some representation of $SL(2, \mathbb{R})$, such a symmetry cannot be directly induced by the isometry group of the spacetime. Let us consider the quantum mechanics of a Rindler particle with spin $s = 0$ and mass $M > 0$. To avoid subtleties involved in the direct Heisenberg-commutation-relations quantization, we determine the one-particle Hilbert space from the general Fock space of the associated quantum field theory. The Klein-Gordon equation for the field φ associated with the particle reads

$$-\partial_t^2 \varphi + \kappa^2 (y \partial_y y \partial_y - y^2 M^2) \varphi = 0. \quad (4)$$

Local wavefunctions of particles are represented by smooth functions φ satisfying (4) enjoying a positive frequency decomposition with respect to the Killing time t . The whole Hilbert space is the completion of the space spanned by those functions with respect to the (positive defined) scalar product

$$(\varphi, \varphi') = i \int_{\Lambda} (\bar{\varphi} \nabla^\mu \varphi' - \varphi' \nabla^\mu \bar{\varphi}) n_\mu d\sigma, \quad (5)$$

Λ being any Cauchy surface with induced metric $d\sigma$ and unit normal vector n pointing toward the future. Referring to the metric (3) and (4), the decomposition in positive-frequency modes of a wavefunction φ reads

$$\varphi(t, y) = \int_0^{+\infty} \frac{\Psi_E(y)}{\sqrt{2E}} e^{-iEt} \hat{\varphi}(E) dE, \quad (6)$$

where, defining the adimensional parameter $\omega = E/\kappa$,

$$\Psi_E(y) = \frac{\sqrt{2\omega \sinh(\pi\omega)}}{\pi} K_{i\omega}(My), \quad (7)$$

K_a being the usual Bessel-McDonald function. Notice that there is no degeneracy in E , any value E admits a unique mode Ψ_E and the modes span the whole Hilbert space. Finally $\Psi_E(y) = \overline{\Psi_E(y)}$. Notice that there is no limit of $K_{i\omega}(My)$ as $M \rightarrow 0$ and for $M = 0$ there are two (complex) modes associated with each value E , but we consider the case $M > 0$ only. The scalar product (5) reads, in terms of functions $\hat{\varphi}$:

$$(\varphi, \varphi') = \int_0^{+\infty} \overline{\hat{\varphi}(E)} \hat{\varphi}'(E) dE. \quad (8)$$

As a consequence the one-particle Hilbert \mathcal{H} space is realized as $L^2(\mathbb{R}^+, dE)$ where dE denotes the usual Lebesgue measure and the one-particle Hamiltonian. H itself is realized as the multiplicative operator

$$(H\hat{\varphi})(E) = E\hat{\varphi}(E),$$

with domain $\mathcal{D}(H) = \{\hat{\varphi} \in L^2(\mathbb{R}^+, dE) \mid \int_{\mathbb{R}^+} |E\hat{\varphi}(E)|^2 dE < \infty\}$. H is self-adjoint on $\mathcal{D}(H)$ with spectrum $\sigma(H) = [0, +\infty)$.

We want to show that the physical system is invariant under the action of unitary representations of the conformal group $SL(2, \mathbb{R})$ and, in fact, \mathcal{H} is nothing but a irreducible representation space. These results are quite remarkable because (a) $SL(2, \mathbb{R})$ is not a background symmetry, in that sense the found symmetry is *hidden*, and (b) we have explicitly assumed that $M \neq 0$ and thus the theory involves a length scale M^{-1} . Actually, the $SL(2, \mathbb{R})$ representation comes out from the Hamiltonian operator of a particle and the scale M turns out to be harmless. Indeed, differently from the Minkowskian case, the minimum of the spectrum of the energy is 0 also if $M > 0$. This is due to the presence of the gravitational energy of a particle which is encompassed by H itself.

Consider the following triple of symmetric differential operators defined on some common invariant and dense subspace $\mathcal{D} \subset L^2(\mathbb{R}^+, dE)$ of smooth functions

$$H_0 = E, \quad (9)$$

$$D = -i \left(\frac{1}{2} + E \frac{d}{dE} \right), \quad (10)$$

$$C = -\frac{d}{dE} E \frac{d}{dE} + \frac{(k - \frac{1}{2})^2}{E}. \quad (11)$$

where $k \in \mathbb{R}$ is a fixed pure number. On \mathcal{D} , it holds

$$[H_0, D] = iH_0, \quad (12)$$

$$[C, D] = -iC, \quad (13)$$

$$[H_0, C] = 2iD. \quad (14)$$

The commutations rules above are those of $sl(2, \mathbb{R})$. Therefore one expects that there is a unitary representation of $SL(2, \mathbb{R})$ in \mathcal{H} obtained by taking the imaginary exponential of self-adjoint extensions of the three operators above. In particular one also expects that H_0, D, C are essentially self-adjoint on \mathcal{D} , in order to have unique self-adjoint extensions, and that the unique self-adjoint extension of H_0 coincides with the Hamiltonian operator H . In fact, all that is true if and only if $k \in \{1/2, 1, 3/2, \dots\}$, but the proof is not straightforward because it involves a very careful analysis of the definition of \mathcal{D} . Some details will be supplied in section 6. Therein we also analyze the interplay between H and the Hamiltonian $-\frac{1}{2} \left(\frac{d^2}{dx^2} + \frac{1}{4x^2} \right)$, which has largely appeared in the literature [10, 11, 8, 12, 1, 13], also to correct some erroneous statements about the spectra of self-adjoint extensions used in some recent works.

From now on we assume that (a) the unitary representation of $SL(2, \mathbb{R})$ exists (in particular it must be $k \in \{1/2, 1, 3/2, \dots\}$ ¹), (b) H_0, C, D are essentially self-adjoint on some dense invariant subspace \mathcal{D} and their self-adjoint extensions are the generators of the representation, (c) $H = H_0$

¹If $k \notin \{1/2, 1, 3/2, \dots\}$ the involved representation concerns the universal covering of $SL(2, \mathbb{R})$. This fact was not noticed in [10].

on \mathcal{D} . The unique self-adjoint extension of C, D will be denoted by the same symbols. Let us pass to consider the time-dependent operators (defined on $\exp(-itH)\mathcal{D}$)

$$D(t) = D + tH, \quad (15)$$

$$C(t) = C + 2tD + t^2H. \quad (16)$$

If X_H denotes the Heisenberg representation of the operator X , using (12), (13), (14), the following commutations rules on \mathcal{D} are trivially proven

$$\delta_D H = i[H, D_H(t)] + \frac{\partial D_H(t)}{\partial t} = 0, \quad (17)$$

$$\delta_C H = i[H, C_H(t)] + \frac{\partial C_H(t)}{\partial t} = 0, \quad (18)$$

$$\delta_H H = [H, H] = 0. \quad (19)$$

The set of those commutation rules is rigorously written

$$e^{-iuX(t)}e^{-itH} = e^{-itH}e^{-iuX(0)}, \quad (20)$$

where $X(t)$ is the self-adjoint extension of any real linear combination of $H(t), C(t), D(t)$ (in Schrödinger picture) and, in our convention, $\exp(-iuX(t))$, $u \in \mathbb{R}$, is the unitary one-parameter subgroup with generator $X(t)$.

The commutation rules above have three straightforward but important consequences, (a) *the physical system is invariant under the unitary group generated by $H, C_H(t), D_H(t)$* , moreover (b) *$H, C_H(t), D_H(t)$ are constants of motion*, finally (c) *at each time $t \in \mathbb{R}$, the unitary groups generated by, respectively $H, C(t), D(t)$ and $H, C_H(t), D_H(t)$ are a unitary representations of $SL(2, \mathbb{R})$ too.*

As further remarkable facts, we stress that (see section 6), (d) for each fixed $k \in \{1/2, 1, 3/2, \dots\}$, $\mathcal{H} = L^2(\mathbb{R}^+, dE)$ turns out to be *irreducible* under the action of the $SL(2, \mathbb{R})$ unitary representation. Moreover, (e) the representation is *faithful* (i.e. injective) if and only if $k = 1/2$.

We finally re-stress that the action of the conformal group is not the usual one which acts on the field operators but the central rôle is played by the Hamiltonian: The representation is realized in the L^2 space associated to the spectral resolution of H . In spite of the presence of the mass of the particles, the spectrum $\sigma(H) = [0, +\infty)$ reveals no explicit physical scale.

3 Hidden conformal symmetry and thermal states in 2D Rindler spacetime.

In the following we analyze some physical consequences of the found hidden conformal representations paying attention to the basic representation $k = 1/2$ in particular. If $k = 1/2$, and only in that case, the physical system gives rise to a faithful irreducible representation of $SL(2, \mathbb{R})$. This simplest case, in a certain sense, is similar to the case of a relativistic spin 1/2 particle when $SL(2, \mathbb{R})$ is replaced by $SL(2, \mathbb{C})$.

The self-adjoint operators in the representation $sl(2, \mathbb{R})$ single out a new algebra of physical observables which are *constants of motion* as a consequence of the conformal invariance of the system. Among these observables, we pick out that represented by,

$$K_\lambda = \frac{1}{2} \left(\frac{\lambda}{\kappa} H + \frac{\kappa}{\lambda} C \right), \quad (21)$$

where $\lambda > 0$ is a pure number and the surface gravity κ has been introduced to make sensible the sum of H and C which have different physical dimensions.

It is worth stressing that our bifurcate Killing horizon gives us a preferred constant κ to put in the definition of K_λ (which does not depend on the mass of the particle). In general there is not such a natural constant suggested by the theory (see [10] and section 6).

Actually K_λ must be defined as a time-dependent observable in order to produce a constant of motion

$$K_\lambda(t) = \frac{1}{2} \left(\frac{\lambda}{\kappa} H + \frac{\kappa}{\lambda} C \right) + \frac{\kappa}{2\lambda} (C + 2tD + t^2H), \quad (22)$$

and $K_\lambda = K_\lambda(0)$. K_λ has been introduced in [10] and considered in several papers because of its appealing properties (e.g., see [8, 9]). K_λ is essentially self-adjoint if defined on \mathcal{D} and remarkably, the spectrum of its self-adjoint extension is purely discrete and does not depend on $\lambda\kappa$ (but it depends on k). It can be proven as follows. If we define the pair of operators

$$A_\pm = \frac{1}{2} \left(\frac{\lambda}{\kappa} H - \frac{\kappa}{\lambda} C \right) \mp iD, \quad (23)$$

the $sl(2, \mathbb{R})$ commutation rules imply

$$[K_\lambda, A_\pm] = \pm A_\pm. \quad (24)$$

Then we look for solutions in $L^2(\mathbb{R}, dE)$ of the couple of equations

$$A_- Z^{(k)} = 0, \quad (25)$$

$$K_\lambda Z^{(k)} = k Z^{(k)}, \quad (26)$$

for some $k \in \mathbb{R}$. If a normalized solution exists, (24) entail that the set vectors recursively defined as $Z_k^{(k)} = Z^{(k)}$ and, for $m = k, k+1, \dots$

$$Z_{m+1}^{(k)} = [m(m+1) - k(k-1)]^{-1/2} A_+ Z_m^{(k)},$$

also satisfy

$$K_\lambda Z_m^{(k)} = m Z_m^{(k)},$$

and thus they are pairwise orthogonal and normalized. Let us consider the simplest case $k = 1/2$. By a direct computation one finds that a set of orthogonal vectors $\{Z_m^{(k)}\}$ exist with the form

$$Z_m^{(k)}(E) = \sqrt{\frac{\Gamma(m-k+1)}{E \Gamma(m+k)}} e^{-\lambda E/\kappa} \left(\frac{2\lambda E}{\kappa} \right)^k L_{m-k}^{(2k-1)} \left(\frac{2\lambda}{\kappa} E \right), \quad (27)$$

and in particular for $k = 1/2$

$$Z_m^{(1/2)}(E) = \sqrt{\frac{2\lambda}{\kappa}} e^{-\lambda E/\kappa} L_{m-1/2} \left(\frac{2\lambda}{\kappa} E \right), \quad (28)$$

where $m = k, k+1, k+2, \dots$, $L_p^{(\beta)}$ are modified Laguerre's polynomials and L_n are Laguerre's polynomials [14]. It is known that, for each $k > 0$ the vectors $Z_m^{(k)}$ define a Hilbert basis of $L^2(\mathbb{R}^+, dE)$. That result suggests to define, for each fixed k , \mathcal{D} as the space finitely spanned by the vectors $Z_m^{(k)}$. In fact this is a correct prescription, in section 6 we give details. (Remind that the representation is a representation of $SL(2, \mathbb{R})$ only if $k \in \{1/2, 1, 3/2, 2, 5/2, \dots\}$ as said above.) As a consequence K_λ and $K_\lambda(t)$ are essentially self-adjoint (on \mathcal{D} and $\exp(-itH)\mathcal{D}$ respectively) and the spectrum of their unique self-adjoint extension is $k, k+1, k+2, \dots$ non depending on t .

The found eigenvectors have a non stationary time evolution, because they are not eigenstates of the Hamiltonian, however, as $K_\lambda(t)$ is a constant of motion $\exp(-itH)Z_m^{(k)}$ is an eigenvector of $K_\lambda(t)$ with the initial eigenvalue m .

Let us analyze the energy content of the base of the $SL(2, \mathbb{R})$ representation in the case $k = 1/2$. The probability density to get the energy value E in the state $Z_m^{(1/2)}$ does not depend on time and reads

$$\rho_m(E) = |Z_m^{(1/2)}(E)|^2 = \beta e^{-\beta E} (L_{m-1/2}(\beta E))^2, \quad (29)$$

where we have introduced the parameter $\beta = \lambda/\kappa > 0$. In particular

$$\rho_0(E) = \beta e^{-\beta E}. \quad (30)$$

It is clear that β can be interpreted as an inverse temperature and $\rho_0(E)$ is nothing but a canonical ensemble distribution at the temperature β^{-1} . The other eigenvalues m give rise to polynomial deformation to that distribution and the canonical ensemble distribution behavior is preserved at leading order as $\beta \rightarrow 0$, $\rho_m(E) \sim C_m \beta e^{-\beta E}$. (This fact does not hold for $k > 1/2$ because $\rho_m(E) \sim C_m \beta^{2k} E^{2k-1} e^{-\beta E}$ as $\beta \rightarrow 0$.)

Despite $Z_{1/2}^{(1/2)}$ being not stationary, it is possible to associate a stationary state with it as follows. Take a suitable observable \mathcal{A} assuming that it does not depend on t . Suppose that the physical system is represented by the state $\Psi_t = \exp(-itH)Z_{1/2}^{(1/2)}$ and one is interested in getting the averaged value of \mathcal{A} within a very long period of time. In other words, one wants to compute

$$\langle \mathcal{A} \rangle = \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T (\Psi_t, \mathcal{A} \Psi_t) dt.$$

Using the explicit expression of $Z_{1/2}^{(1/2)}$,

$$\langle \mathcal{A} \rangle = \lim_{T \rightarrow +\infty} \frac{\beta}{2T} \int_{-T}^T \int_0^\infty dE \int_0^\infty dE' e^{-\beta(E+E')/2} e^{-it(E-E')} \langle E | \mathcal{A} | E' \rangle.$$

The limit can be computed by regularizing the continuous spectrum by means of a discrete set of values and finally restoring the continuous extent by means of a normalization factor N_β . By that way one gets

$$\langle \mathcal{A} \rangle = \frac{1}{N_\beta} \int_0^\infty e^{-\beta E} \langle E | \mathcal{A} | E \rangle dE = \text{Tr}(\rho_\beta \mathcal{A}) ,$$

with

$$\rho_\beta = \frac{1}{N_\beta} \int_0^\infty dE e^{-\beta E} |E\rangle \langle E| .$$

Actually ρ_β is not trace class because $N_\beta \sim \delta(0)\beta^{-1}$ and the integral in front to N_β^{-1} diverges too. Therefore the expression above must be understood in the sense of the regularization and referred to a suitable class of observables including functions of H . As is well-known, among the set of values of β , there is a preferred value $\beta^{-1} = \frac{\kappa}{2\pi}$, the Hawking(-Unruh-Fulling) inverse temperature, corresponding, in our approximation, to the thermal equilibrium temperature of the particle with a bifurcate black hole. That value determines a preferred operator K_λ .

4 Non bifurcate Killing horizon and manifest $SL(2, \mathbb{R})$ invariance.

In this section we analyze a massive free particle propagating in a portion of AdS_2 spacetime. As is well-known, dropping the angular part, that spacetime is a near-horizon approximation of an extremal Reissner-Nördstrom black hole. Here by AdS_2 we mean the *universal covering* of the Lorentzian manifold which is properly called “Anti-de Sitter spacetime”. This is the way one usually follows to get rid of the presence of closed timelike paths. In our case, the Killing horizon is not bifurcate differently from the Schwarzschild case. Moreover, the spacetime is not globally hyperbolic and thus quantum field theory needs much care to be defined. However we do not deal with these subtleties here. With an appropriate choice of the Klein-Gordon modes, once again the spectral Hamiltonian representation of a particle space reveals a $SL(2, \mathbb{R})$ symmetry. However the AdS_2 case physically differs from Rindler one due to some new features. Now the background geometry selects a distinguished value for k which depends on the mass of the particle. On the other hand the thermal spectrum of modes is not allowed among these selected representations because one finds $k > 1/2$ no matter the value of the mass M . We start by writing a local metric for AdS_2

$$ds^2 = -\frac{x^2}{\ell^2} dt^2 + \frac{\ell^2}{x^2} dx^2 , \quad (31)$$

ℓ^2 being related to the cosmological constant and $t \in \mathbb{R}$, $x \in (0, +\infty)$. This metric is defined in a portion of AdS_2 spacetime which plays the analogous rôle as the Rindler wedge in Minkowski spacetime. Defining $r = \ell^2/x$ the metric above becomes Robinson-Bertotti’s metric

$$ds_{RB}^2 = \ell^2 \frac{-dt^2 + dr^2}{r^2} , \quad (32)$$

The whole AdS_2 spacetime is represented by the vertical stripe in fig.1 where Robinson Bertotti's metric is valid in each region indicated by R-B and delimited by diagonal lines corresponding to $t = \pm\infty$. In the figure $U = T + R$ and $W = R - T$.

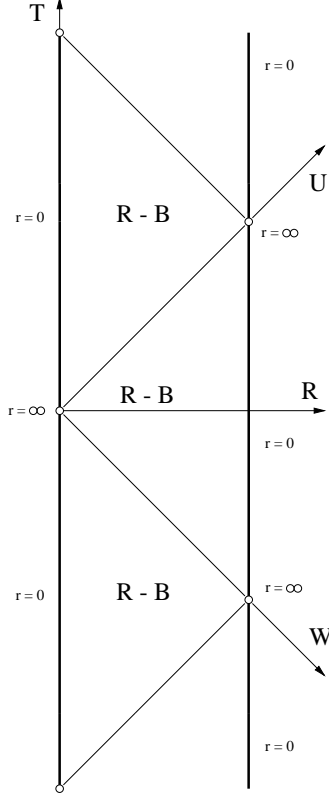


figure 1

The field φ , describing a free particle with mass M propagating in a R-B region, satisfies the Klein-Gordon equation:

$$-\ell^2 \partial_t^2 \varphi + (x^2 \partial_x x^2 \partial_x - M^2 x^2) \varphi = 0 . \quad (33)$$

As usual, φ can be decomposed in stationary modes

$$\varphi(t, x) = \int_0^{+\infty} \frac{\Psi_E(x)}{\sqrt{2E}} e^{-iEt} \hat{\varphi}(E) dE . \quad (34)$$

We chose the set of modes, solutions of the K-G equation above,

$$\Psi_E(x) := J_\nu \left(-\frac{\ell^2 E}{x} \right) \sqrt{\frac{\ell^2 E}{x}} \quad (35)$$

where $\nu = \sqrt{1/4 + (M/\ell)^2}$, $J_\nu(y)$ being a Bessel function. These modes give rise to a complete spectral measure on $L^2(\mathbb{R}^+, dE)$ and, as the metric is static, this is sufficient to build up a quantum field theory regardless the spacetime is not globally hyperbolic [15, 16]. As in the Rindler spacetime case, for every value of $E \in \mathbb{R}^+$ there is a unique mode $\Psi_E(x)$. The one-particle Hamiltonian H is realized as the self-adjoint multiplicative operator over wavefunctions $\hat{\varphi} = \hat{\varphi}(E)$ in $L^2(\sigma(H), dE)$ and $\sigma(H) = [0, \infty)$. The scalar product in $L^2(\mathbb{R}^+, dE)$ coincides with the scalar product (5) performed with respect to the corresponding wavefunctions in the left-hand side of (34) if Λ is any $t = \text{constant}$ surface. Then, exactly as in the Rindler case, the physical system turns out to be invariant under the irreducible unitary representation of $SL(2, \mathbb{R})$ generated by generators (9), (10), (11) with k fixed in $\{1/2, 1, 3/2, \dots\}$. (If $0 < k \notin \{1/2, 1, 3/2, \dots\}$ the irreducible representation concerns the universal covering of $SL(2, \mathbb{R})$.) As before, the Hilbert space of the system coincides with such an irreducible representation space. On the other hand, the elements ω of the universal covering of $SL(2, \mathbb{R})$ can be represented as a group of isometries T_ω of AdS_2 . In particular the time evolution is one of the isometries of the group. A basis of Killing vector fields whose integral lines define the group of isometries is

$$h = \frac{\partial}{\partial t}, \quad d := t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x}, \quad c = \left(t^2 + \frac{\ell^4}{x^2}\right) \frac{\partial}{\partial t} - 2tx \frac{\partial}{\partial x}. \quad (36)$$

In fact, it is a trivial task to show that these fields are Killing fields and their Lie algebra is $sl(2, \mathbb{R})$. These vector fields can also be interpreted as generators of a group of automorphisms α_ω of the algebra of the fields:

$$\alpha_\omega(\varphi)(t, x) = \varphi(T_\omega(t, x)).$$

Beyond the equation of motion, these automorphism preserve the scalar product (5) because they are induced by isometries and thus define a unitary representation of the universal covering of $SL(2, \mathbb{R})$ on $L^2(\mathbb{R}^+, dE)$. Using the decomposition (34), the generators of that unitary $SL(2, \mathbb{R})$ representation turn out to be associated with the generators of the isometries as follows

$$h \leftrightarrow H = E, \quad (37)$$

$$d \leftrightarrow D(t) = tE - i \left(\frac{1}{2} + E \frac{d}{dE} \right), \quad (38)$$

$$c \leftrightarrow C(t) = -\frac{d}{dE} E \frac{d}{dE} + \frac{\left[\frac{1}{4} + \left(\frac{M}{\ell} \right)^2 \right]}{E} - 2it \left(\frac{1}{2} + E \frac{d}{dE} \right) + t^2 E. \quad (39)$$

We have found that, depending on the value of ℓ and M , the geometry picks out one of energy irreducible representations of $SL(2, \mathbb{R})$ (or its universal covering) found in section 2. $k > 0$ is uniquely determined by $(k - 1/2)^2 = 1/4 + (M/\ell)^2$ and the value $k = 1/2$ is not allowable for real values of M . The massless case determines the least (semi)integer value of k , $k = 1$. As a consequence, it is not possible to get a thermal energy spectrum from the vectors (also as $2\lambda\ell = \beta \rightarrow 0$)

$$Z_m^{(k)}(E) = \sqrt{\frac{\Gamma(m - k + 1)}{E \Gamma(m + k)}} e^{-\lambda E} (2\lambda\ell E)^k L_{m-k}^{(2k-1)}(2\lambda\ell E), \quad (40)$$

which are the complete set of orthonormal eigenvectors of the operator

$$K_\lambda = \frac{1}{2} \left(\lambda \ell H + \frac{C}{\lambda \ell} \right) . \quad (41)$$

The energy distribution for $M = 0$ and $m = k$ now reads

$$\rho_k(E) = |Z_1^{(1)}(E)|^2 = (2\lambda\ell)^2 E e^{-2\lambda\ell E} . \quad (42)$$

Everything we said is referred to the $SL(2, \mathbb{R})$ representations induced by the geometry. One could wonder if it could make sense to consider proper hidden $SL(2, \mathbb{R})$ representations in AdS_2 spacetime. In that case, the representation with $k = 1/2$ would present the same thermal features found in the bifurcate horizon case. However, we remind the reader that, differently from the bifurcate horizon case, there is no preferred nonvanishing value for the temperature of the thermal states of the field if the horizon is nonbifurcate [17, 18, 19].

5 Manifest $SL(2, \mathbb{R})$ symmetry and global time evolution in AdS_2 spacetime.

It is known [20] that, at least for $M = 0$, the Wightman function of the vacuum state referred to the Killing time t and defined in a B-R region, can be analytically extended in the whole AdS_2 spacetime. The found global Wightman function turns out to coincide with that built up with respect to the vacuum state referred to a global Killing time (indicated by T in the figure). This fact leads us to investigate about the global behavior of the quantum system in AdS_2 spacetime. Following that way, a dynamical interpretation of the operator K_λ for a particular value of λ arises. As suggested in [8, 9] K_λ can be seen as a Hamiltonian referred to a different time coordinate in the spacetime. The machinery developed in this paper enable us to produce a precise statement of that fact. As said above AdS_2^2 can be equipped with global coordinates R, T and the global metric reads

$$ds^2 = \frac{1}{\sin^2(R/\ell)} (-dT^2 + dR^2) , \quad (43)$$

where $T \in \mathbb{R}$ and $R \in (0, \pi\ell)$. T is a distinguished global Killing time different from the local Killing time t defined in each R-B patch. The relationship between local and global coordinates in the R-B patch containing part of the axis R (see figure) reads (e.g., see [20])

$$\frac{\ell}{x} + \frac{t}{\ell} = \cot \left(\frac{R - T}{2\ell} \right) , \quad (44)$$

$$\frac{\ell}{x} - \frac{t}{\ell} = \cot \left(\frac{R + T}{2\ell} \right) . \quad (45)$$

²Actually, we mean the universal covering of the proper AdS_2 spacetime.

In the new global coordinates the Killing field $\ell h/2 + c/(2\lambda\ell)$, corresponding to the generator $K_\lambda(t)$ of the unitary representation of (the universal covering of) $SL(2, \mathbb{R})$ with $\lambda = 1$, reads

$$\frac{1}{2} \left(\ell h + \frac{c}{\ell} \right) = -\frac{1}{\ell} \frac{\partial}{\partial T}, \quad (46)$$

As a consequence we conclude that the operator $H' = \ell^{-1}K_1(0)$ is nothing but the Hamiltonian of a particle with respect to the global Killing time T . Equivalently, H' is the generator of the past displacements along the global time T . Notice that H' is defined in the same Hilbert space \mathcal{H} and gives rise to a unitary evolutor $\exp(-iTH')$ for $T \in (-\infty, +\infty)$. This apparently unexpected result is consequence of the fact that the surfaces $t = 0$ and $T = 0$ coincide and can be used to define the Hilbert space of the solution of K-G equation (despite the surface being not Cauchy). A few words concerning $H'(t) = \ell^{-1}K_1(t)$ are in order. If $\Psi_v \in \mathcal{H}$ is given at $t = v$ and corresponds to Ψ given at $t = 0$ along the t temporal evolution, $\exp(-iuH'(v))\Psi_v = \exp(-ivH)\exp(-iuH')\Psi$ denotes the state obtained by the t evolution up to the time v of the state at $t = 0$, $\exp(-iuH')\Psi$, obtained by a past displacement along the time T of Ψ .

6 $SL(2, \mathbb{R})$ unitary representations and operators $\frac{1}{2} \left(-\frac{d^2}{dx^2} + \frac{g}{x^2} \right)$.

The family of unitary irreducible $SL(2, \mathbb{R})$ representations are well known (see [21, 22] for a complete treatment). The unitary *positive energy irreducible representations* (i.e. those where the operator corresponding to H , in the realization of $sl(2, \mathbb{R})$ studied in this work, has positive spectrum) are labeled by the values $k = 1/2, 1, 3/2, \dots$. k determines each representation up to a unitary equivalence. A representation as well as its representation Hilbert space is denoted by D^k . The family of all D^k is called the “discrete series” [23, 24]. The unitary representations of the universal covering group of $SL(2, \mathbb{R})$ with a lowest weight are similar and, once again, are indicated with D^k with $k \in (0, +\infty) \setminus \{1/2, 1, 3/2, \dots\}$. All those representations are encompassed in a well-known model [10] we go to illustrate. Let $x \in \mathbb{R}^+$ (whose dimensions are $[M]^{-1/2} = [L]^{1/2}$) be a field in dimension $d = 1$, with Lagrangian $L(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{g}{2x^2}$. Above g is an adimensional constant. Consider the action of $SL(2, \mathbb{R})$ on \mathbb{R} given by

$$\omega : t \mapsto t' = \frac{a t + c}{b t + d},$$

where t and t' indicates instants of time and the matrix $\omega \in SL(2, \mathbb{R})$ admits a b as the former row c d as the latter row. That nonlinear representation induces a conformal transformation of the field $x = x(t)$ under $\omega \in SL(2, \mathbb{R})$,

$$x(t) \mapsto x'(t') = (b t + d)^{-1} x(t).$$

This transformation preserves the action (but not the Lagrangian) of the field x . To consider the quantum version of the story, let us introduce the operator $X_H(t)$ (denoted by $Q(t)$ in [10]) on $L^2(\mathbb{R}^+, dx)$ which represents the quantum field operator associated with x in the Heisenberg

picture, the Schrödinger picture being $(X\psi)(x, t) = x\psi(x, t)$. Then the unitary implementation of the action of $SL(2, \mathbb{R})$ above must have the form

$$X'_H(t) = U(\omega) X_H(t) U(\omega)^\dagger ,$$

where $\omega \mapsto U(\omega)$ is a unitary representation. For $g \geq -1/4$, a formal realization of the generators of $\{U(\omega)\}_{\omega \in SL(2, \mathbb{R})}$ on $L^2(\mathbb{R}^+, dx)$ is given as follows. Consider the three symmetric differential operators [10] defined on some dense invariant domain of smooth functions $\tilde{\mathcal{D}} \subset L^2(\mathbb{R}^+, dx)$,

$$\tilde{H} = \frac{1}{2} \left(-\frac{d^2}{dx^2} + \frac{g}{x^2} \right) , \quad (47)$$

$$\tilde{D} = \frac{i}{2} \left(\frac{1}{2} + x \frac{d}{dx} \right) , \quad (48)$$

$$\tilde{C} = \frac{x^2}{2} . \quad (49)$$

These operators define the so-called DFF model. Notice that, barring problems with self-adjoint extensions, \tilde{H} must be considered the Hamiltonian operator of the system. That operator has been largely studied in the literature on conformal invariance in black hole backgrounds [11, 8, 12, 1, 13]. It is a trivial task to show that the three operators above satisfy the commutation rules of $sl(2, \mathbb{R})$ on $\tilde{\mathcal{D}}$. Then define time-dependent operators $\tilde{D}(t), \tilde{C}(t)$ similarly to (15) and (16) and pass to the Heisenberg picture $\tilde{H}_H = \tilde{H}, \tilde{C}_H(t), \tilde{D}_H(t)$. Following the same way as in section 2 one expects that the system should be invariant under a unitary representation of $SL(2, \mathbb{R})$, $\{U(\omega)\}_{\omega \in SL(2, \mathbb{R})}$, and self-adjoint extensions of these operators should be the generators of the representation. In the following we sketch some proofs of these facts also discussing some subtleties concerning the definition of $\tilde{\mathcal{D}}$ ³, the spectra of self-adjoint extensions of \tilde{H} and correcting some statements used in the literature [11].

Before to start with the analysis we stress that, concerning $SL(2, \mathbb{R})$ representations, everything proven for the realization $\tilde{H}, \tilde{D}, \tilde{C}$ generalize to the realization H, D, C considered in section 3 by means of the following unitary equivalence. Consider the unitary transformation $U : L^2(\mathbb{R}^+, dE) \rightarrow L^2(\mathbb{R}^+, dx)$ induced by the densely defined transformation which preserves the scalar product

$$\psi(x) = \int_0^{+\infty} \sqrt{x} J_{\sqrt{g+\frac{1}{4}}}(\sqrt{2E} x) \hat{\varphi}(E) dE .$$

with densely defined inverse:

$$\hat{\varphi}(E) = \int_0^{+\infty} \sqrt{x} J_{\sqrt{g+\frac{1}{4}}}(\sqrt{2E} x) \psi(x) dx .$$

Under that unitary transformation we get

$$\tilde{H} = U H_0 U^{-1} , \quad (50)$$

$$\tilde{D} = U D U^{-1} , \quad (51)$$

$$\tilde{C} = U C U^{-1} , \quad (52)$$

³These proofs and a discussion on $\tilde{\mathcal{D}}$ do not appear in [10].

where the operators in the left hand side are defined on $\tilde{\mathcal{D}}$ and H_0, D, C are those introduced in section 2 and defined on \mathcal{D} which now can properly be defined $\mathcal{D} = U^{-1}\tilde{\mathcal{D}}$. The parameter k which appears in the definition of C is related with g as follows

$$k(g) = \frac{1}{2} \left(1 + \sqrt{g + \frac{1}{4}} \right) \quad \text{with } g \geq -1/4.$$

To go on, let us examine the unitary $SL(2, \mathbb{R})$ representation in details.

Existence of unitary $SL(2, \mathbb{R})$ representations. A known result by Nelson (Corollary 9.1, Lemma 5.2 in [25]) implies that if the symmetric operator $\tilde{H}^2 + \tilde{D}^2 + \tilde{C}^2$ is essentially self-adjoint in a dense invariant linear space $\tilde{\mathcal{D}}$, then $\tilde{H}, \tilde{D}, \tilde{C}$ are essentially self-adjoint on $\tilde{\mathcal{D}}$ and their self-adjoint extensions generate a unitary representation of the simply connected Lie group associated with $sl(2, \mathbb{R})$, i.e. the universal covering of $SL(2, \mathbb{R})$. Such a unitary representation preserves the one-parameters subgroups generated by the elements of the Lie algebra which become subgroups generated by the associated self-adjoint operators.

To use Nelson's result, consider the Hilbert basis of $L^2(\mathbb{R}^+, dx)$, with $m = k(g), k(g) + 1, k(g) + 2, \dots$

$$\tilde{Z}_m^{(k(g))}(x) = \sqrt{\frac{2\Gamma(m - k(g) + 1)}{x\Gamma(m + k(g))}} \left(\frac{x^2}{\beta} \right)^{k(g)} e^{-\frac{x^2}{2\beta}} L_{m-k(g)}^{2k(g)-1} \left(\frac{x^2}{\beta} \right), \quad (53)$$

in particular, if $g = -1/4$, $k(g) = 1/2$ and

$$\tilde{Z}_m^{(1/2)}(x) = \sqrt{\frac{2x}{\beta}} e^{-\frac{x^2}{2\beta}} L_{m-1/2} \left(\frac{x^2}{\beta} \right). \quad (54)$$

$L_n^{(\alpha)}$ are the modified Laguerre polynomial of order n , $L_n^{(0)} = L_n$ are Laguerre's polynomials. Similarly to that found in section 3, these functions are eigenfunctions with eigenvalue m of the differential operator

$$\tilde{K}_\beta = \frac{1}{2} \left(\beta \tilde{H} + \frac{\tilde{C}}{\beta} \right) = -\frac{\beta}{4} \frac{d^2}{dx^2} + \frac{\beta g}{4x^2} + \frac{x^2}{4\beta}, \quad (55)$$

β being any positive constant with $[\beta] = [L]$ which, differently from the Rindler space model, is not supplied by the DFF model itself. Using the operators A_\pm introduced in section 3 and a Casimir operator, it is possible to show that the vectors $\tilde{Z}_m^{(k(g))}$ define a set of analytic vectors of the operator $\tilde{H}^2 + \tilde{D}^2 + \tilde{C}^2$. As a consequence if $\tilde{\mathcal{D}}$ is defined as the linear space finitely spanned by the vectors $\tilde{Z}_m^{(k(g))}$, \tilde{K}_β is essentially self-adjoint on $\tilde{\mathcal{D}}$ and the spectrum of its self-adjoint extension is $\{k(g), k(g) + 1, k(g) + 2, \dots\}$. Moreover Nelson's results entail that the self-adjoint extensions of $\tilde{H}, \tilde{C}, \tilde{D}$ generate a unitary representation of the universal covering of $SL(2, \mathbb{R})$. $SL(2, \mathbb{R})$ does not coincide with its universal covering because it is not simply connected it being homeomorphic to $S^1 \times \mathbb{R}^2$. However if $k(g) \in \{1/2, 1, 3/2, \dots\}$ and only in that case, it is

possible to conclude that the found representation is, in fact, a representation of $SL(2, \mathbb{R})$ too. This fact was not considered in [10] where $2k$ is not supposed to assume integer values only⁴.

Faithful representations. A $SL(2, \mathbb{R})$ decomposition rule holds as a direct consequence of polar decomposition theorem. For every $\omega \in SL(2, \mathbb{R})$,

$$\omega = R(\theta_\omega)E(\chi_\omega)R(\theta'_\omega),$$

where $\theta_\omega, \theta'_\omega \in [0, 2\pi)$, $\chi \in \mathbb{R}$, $R(\alpha) \in SL(2, \mathbb{R})$ is a pure rotation corresponding to $\exp(i2\theta\tilde{K}_\beta)$ and $E(\chi_\omega) = \text{diag}(e^\chi, e^{-\chi}) \in SL(2, \mathbb{R})$ is a pure dilatation corresponding to $\exp(i\chi\tilde{D})$.

Using that decomposition rule together with the one-parameter subgroup preservation property of the unitary representation and the (π/k) -periodicity of $\theta \mapsto \exp(i2\theta\tilde{K}_\beta)$ one gets two relevant results. (a) If $k(g) \in \{1, 3/2, 2, \dots\}$ the associated $SL(2, \mathbb{R})$ representation cannot be faithful because $R(\pi/k(g)) \neq I$ but $\exp[i(\pi/k(g))2\tilde{K}_\beta] = I$. Conversely, (b) if $k(g) = 1/2$ (i.e. $g = -1/4$) the representation is faithful.

Irreducible representations. It is possible to show that every unitary $SL(2, \mathbb{R})$ representation found above is irreducible. The proof is based on the following remarks. If an orthogonal projector $P \neq 0$ commutes with the representation, it must commute with $\exp(i\theta\tilde{K}_\beta)$ for all $\theta \in \mathbb{R}$. This implies that P commutes with the projector spectral measure of K_β . As a consequence $P = \sum_{m \in M} |\tilde{Z}_m^{(k(g))}\rangle \langle \tilde{Z}_m^{(k(g))}|$ for some $M \subset I = \{k(g), k(g) + 1, k(g) + 2, \dots\}$. The invariant subspace $L = P(L^2(\mathbb{R}^+, dx))$ admits the Hilbert basis of vectors $\tilde{Z}_m^{(k(g))}$ with $m \in M$. However, if $m \in M$ and $n \notin M$ it must be $(\tilde{Z}_m^{(k(g))}, \exp(it\tilde{H})\tilde{Z}_n^{(k(g))}) \neq 0$ for some $t \in \mathbb{R}$. If not, using a Fourier transformation we could conclude that $\tilde{Z}_m^{(k(g))}(E)\tilde{Z}_n^{(k(g))}(E) = 0$ for all $E \in \mathbb{R}^+$ which is not true. As $\exp(it\tilde{H})$ is an element of the representation, the space L can be invariant only if $M = I$ and thus $L = L^2(\mathbb{R}^+, dx)$.

We conclude that $L^2(\mathbb{R}^+, dx)$ and the representation generated by the self-adjoint extensions of $\tilde{H}, \tilde{C}, \tilde{D}$ define an irreducible unitary representation of $SL(2, \mathbb{R})$.

Coming back to the operators H_0, C, D considered in section 2, we notice that $\mathcal{D} = U^{-1}\tilde{\mathcal{D}}$ is nothing but the linear space spanned by the vectors $Z_m^{(k(g))}$ (27) provided $\beta = \lambda\kappa$. These vectors satisfy the constraint $\int_0^{+\infty} E^2 |Z_m^{(k(g))}(E)|^2 dE < \infty$ and $H_0 = U\tilde{H}U^{-1}$ is essentially self-adjoint on \mathcal{D} by construction. As a consequence the unique self-adjoint extension of H_0 , H , is just that defined on the linear space $\mathcal{D}(H) = \{\psi \in L^2(\mathbb{R}^+, dE) \mid \int_{\mathbb{R}^+} E^2 |\psi(E)|^2 dE < \infty\}$. In other words H is just the Hamiltonian in the Rindler space as assumed in section 2. As a consequence all the found unitary representation of $SL(2, \mathbb{R})$ are *positive energy representations*. That is all concerning the problem of the existence, the features of positive energy unitary representations

⁴Roughly speaking the reason of the constraint above is that follows. $SL(2, \mathbb{R})$ does not coincide with the universal covering. The problem arises by the one parameter 2π -periodic subgroup given by the matrices of $R(\theta) \in SO(2) \subset SL(2, \mathbb{R})$ whose generator just corresponds to the operator $2\tilde{K}_\beta$ in the unitary representation we have found. However, if and only if $k(g) \in \{1/2, 1, 3/2, \dots\}$, $\theta \mapsto \exp(i2\theta\tilde{K}_\beta)$ is also 2π -periodic as a consequence of the spectrum found for $2\tilde{K}_\beta$ and thus the found unitary representation is also a unitary representation of $SL(2, \mathbb{R})$ itself.

of $SL(2, \mathbb{R})$ and the assumptions made in section 2 which are proven now.

As a general final general comment, we notice that $\tilde{H}_\beta = \beta \tilde{H}$, $\tilde{C}_\beta = \beta^{-1} \tilde{C}$ and $D_\beta = D$ satisfy the $sl(2, \mathbb{R})$ commutation relations if $\beta > 0$ is a constant with $[\beta] = [L]$. Moreover the unitary transformation

$$(V\psi)(x') = \int_0^{+\infty} \frac{\sqrt{xx'}}{\beta} J_{\sqrt{g+\frac{1}{4}}} \left(\frac{xx'}{\beta} \right) \psi(x) dx$$

interchanges the rôle of \tilde{H}_β and \tilde{C}_β preserving the commutation relations:

$$V\tilde{H}_\beta V^\dagger = \tilde{C}_\beta, \quad (56)$$

$$V\tilde{C}_\beta V^\dagger = \tilde{H}_\beta, \quad (57)$$

$$V\tilde{D}_\beta V^\dagger = -\tilde{D}_\beta. \quad (58)$$

As a consequence:

$$V\tilde{K}_\beta = \tilde{K}_\beta V.$$

A similar transformation can be built up for the $sl(2, \mathbb{R})$ realization in terms of H, C, D composing V and U .

To conclude, we want to focus attention on the self-adjoint extensions of the differential operator \tilde{H} when $g = -1/4$. It is known that $\tilde{H} = -\frac{1}{2} \left(\frac{d^2}{dx^2} + \frac{1}{4x^2} \right)$ on $L^2(\mathbb{R}^+, dx)$ is not essentially self-adjoint [26] on natural domains as $C_0^k(0, +\infty)$, $2 \leq k \leq +\infty$.

We stress that \tilde{H} is essentially self-adjoint in $\tilde{\mathcal{D}}$ as pointed out above and thus no subtleties concerning the self-adjoint generators of $SL(2, \mathbb{R})$ arise by that way.

However, we want to spend a few words on this topic of \tilde{H} because the analysis of the spectrum of the different self-adjoint extensions presented or used in some papers [11, 8, 12, 1] is not correct and part of consequent physical results could not make sense.

Consider the densely defined symmetric operator \tilde{H} as a proper differential operator on a suitable domain $\mathcal{D}(\tilde{H})$. For instance $\mathcal{D}(\tilde{H})$ can be taken as the dense subspace of smooth complex functions with support in $(0, +\infty)$, but a different choice for the domain, as that considered in [26], gives the same class of self-adjoint extensions. The defect indices of the symmetric closed operator \tilde{H}^\dagger are $(1, 1)$ and thus there is a one-parameter class of self-adjoint extensions of \tilde{H}^\dagger (and of \tilde{H} since \tilde{H}^\dagger extends \tilde{H}). The defect spaces \mathcal{D}_+ , \mathcal{D}_- are respectively generated by the square-integrable modified Bessel functions [14] $f_{+i}(x) = \sqrt{x l_0^{-1/2}} H_0^{(1)}(e^{i\pi/4} x / \sqrt{l_0})$ which corresponds to the eigenvalue i/l_0 , and $f_{-i}(x) = \sqrt{x l_0^{-1/2}} H_0^{(2)}(e^{-i\pi/4} x / \sqrt{l_0})$ which corresponds to the eigenvalue $-i/l_0$. l_0 is the used length scale. Following [27, 26], the domain of $\tilde{H}^{\dagger\dagger}$, $\mathcal{D}(\tilde{H}^{\dagger\dagger})$ can be decomposed as

$$\mathcal{D}(\tilde{H}^{\dagger\dagger}) = \mathcal{D}(\tilde{H}^\dagger) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-. \quad (59)$$

The direct sum is not orthogonal. $\tilde{H}^{\dagger\dagger}$ reduces to, respectively, $f \mapsto \pm(i/l_0)f$ on \mathcal{D}_{\pm} and to \tilde{H}^{\dagger} on $\mathcal{D}(\tilde{H}^{\dagger})$. Then, every self-adjoint extension of \tilde{H}^{\dagger} , \tilde{H}_{θ} is obtained by restricting $\tilde{H}^{\dagger\dagger}$ to each domain

$$\mathcal{D}_{\theta} = \{f \in \mathcal{D}(\tilde{H}^{\dagger\dagger}) \mid \lim_{x \rightarrow 0} (\overline{f'_{\theta}(x)}f(x) - \overline{f_{\theta}(x)}f'(x)) = 0\}, \quad (60)$$

where the functions f_{θ} are defined as

$$f_{\theta} = e^{-i\theta/2}f_{+i} + e^{+i\theta/2}f_{-i},$$

for $\theta \in [0, 2\pi)$. (Notice that $f_{\theta} \in C^{\infty}((0, +\infty))$ and the derivative of $f \in \mathcal{D}(\tilde{H}^{\dagger})$ is absolutely continuous and thus (60) makes sense taking (59) into account.) To use (60) it is necessary to know the behavior of $f_{\theta}(x)$ for $x \sim 0$. One has

$$\begin{aligned} (xl_0^{-1/2})^{-1/2}f_{\theta}(x) &= e^{-i\theta/2}J_0(e^{i\pi/4}x/\sqrt{l_0}) + ie^{-i\theta/2}N_0(e^{i\pi/4}x/\sqrt{l_0}) \\ &+ e^{i\theta/2}J_0(e^{-i\pi/4}x/\sqrt{l_0}) - ie^{i\theta/2}N_0(e^{-i\pi/4}x/\sqrt{l_0}) \end{aligned} \quad (61)$$

and thus

$$f_{\theta}(x) \sim 2\sqrt{x l_0^{-1/2}} \left[\cos \frac{\theta}{2} + \frac{2\gamma}{\pi} \sin \frac{\theta}{2} + \frac{ie^{-i\theta/2}}{\pi} \ln \left(\frac{e^{i\pi/4}x}{2\sqrt{l_0}} \right) - \frac{ie^{i\theta/2}}{\pi} \ln \left(\frac{e^{-i\pi/4}x}{2\sqrt{l_0}} \right) \right], \quad (62)$$

$$f'_{\theta}(x) \sim \frac{1}{\sqrt{l_0^{1/2}x}} \left[\cos \frac{\theta}{2} + \frac{2}{\pi}(\gamma + 2) \sin \frac{\theta}{2} + \frac{ie^{-i\theta/2}}{\pi} \ln \left(\frac{e^{i\pi/4}x}{2\sqrt{l_0}} \right) - \frac{ie^{i\theta/2}}{\pi} \ln \left(\frac{e^{-i\pi/4}x}{2\sqrt{l_0}} \right) \right] \quad (63)$$

The function \ln arises from the expansion of N_0 . As it acts on complex numbers it could be interpreted as a *multivalued function*. However $f_{\theta} \in L^2(\mathbb{R}^+, dx)$ and this fact fixes the interpretation of \ln . Indeed separately, $\sqrt{x}J_0(e^{\pm i\pi/4}x/\sqrt{l_0})$ and $\sqrt{x}N_0(e^{\pm i\pi/4}x/\sqrt{l_0})$ do not belong to $L^2(\mathbb{R}^+, dx)$ because of their bad behavior at infinity. However, *if and only if the function \ln is interpreted as a one-valued function (with domain cut along the real negative axis)*, the linear combination of both J_0 and N_0 used above belongs to L^2 . A different interpretation of \ln gives rise to further added terms containing N_0 only and the obtained function cannot belong to L^2 . So, in checking (60) one has to interpret \ln in (62) and (63) as a one-valued function.

Taking that remark into account one sees that, for each H_{θ} with $\theta \neq 0$ there is *exactly one* proper eigenvector in \mathcal{D}_{θ} ,

$$\Psi_{\theta}(x) = C_{\theta}K_0(\sqrt{-E_{\theta}}x)$$

with eigenvalue

$$E_{\theta} = -l_0^{-1}e^{\frac{\pi}{2}\cot \frac{\theta}{2}}.$$

C_{θ} is a normalization constant. The other eigenvectors found in the literature ([11]) actually do not exist. (As a consequence the associated eigenvalues do not exist too.) They have been found because of the multi-valued interpretation of the logarithm which, actually, cannot take place as remarked above, so part of physical results presented in [11, 8, 12, 1] could not make

sense⁵.

If $\theta = 0 \equiv 2\pi$, there are no proper eigenvectors for \tilde{H}_θ and $\sigma(\tilde{H}_{\theta=0}) = [0, +\infty)$. That is the self-adjoint extension of \tilde{H} used above to build up the unitary $SL(2, \mathbb{R})$ representation. As a check one can verify that the functions $\tilde{Z}_m^{(1/2)}$ defining $\tilde{\mathcal{D}}$ satisfy $\left(\overline{f_0} \tilde{Z}_m^{(1/2)} - \overline{f_0} \tilde{Z}_m^{(1/2)'}\right)(x)|_{x \rightarrow 0} = 0$.

7 Discussion, overview and open problems.

Within this paper we have shown that simple physical systems given by massive quantum particles moving in a two-dimensional spacetime which approximates some black hole background, give naturally rise to unitary irreducible representations of $SL(2, \mathbb{R})$ (or its universal covering). In other words these systems are elementary with respect to the conformal symmetry. That symmetry embodies the time evolution of the system. We want to stress that such a result is not trivial at all. For instance consider a massless particle in 2D Rindler space. In that case, differently from the massive case the set of modes associated with a value $E \in \sigma(H) = [0, +\infty)$ is twofold

$$\Psi_E(y) = \frac{1}{\sqrt{2\pi}} e^{\pm i\omega \ln(y\sqrt{\kappa})}, \quad (64)$$

where, as usual, $\omega = E/\kappa$. Therefore, the Hilbert space of a particle is $L^2(\mathbb{R}^+, dE) \otimes \mathbb{C}^2$. In other words, it is the $SL(2, \mathbb{C})$ *reducible* space $D^{1/2} \otimes \mathbb{C}^2$. These particle cannot be considered as elementary systems with respect to $SL(2, \mathbb{R})$. Another interesting example of a non elementary system with respect to $SL(2, \mathbb{R})$ is obtained by formally putting $g = 0$ in the representation class considered in section 6 and extending the Hilbert space from $L^2(\mathbb{R}^+, dx)$ to $L^2(\mathbb{R}, dx)$. In that case, the formal generators (47) and (49) take the form $-\frac{1}{2} \frac{d^2}{dx^2}$ and $\frac{x^2}{2}$. As a consequence, putting $x = \sqrt{m}z$, where m is a constant with the dimensions of a mass, and defining $k = \frac{m}{\beta^2}$, the operator $2\beta^{-1}K_\beta$ reads:

$$-\frac{1}{2m} \frac{d^2}{dz^2} + \frac{kz^2}{2}.$$

This is the Hamiltonian of a harmonic oscillator. The eigenvalues of K_β are $\{1/4, 3/4, 5/4, \dots\}$ and thus the space of the system cannot coincide with an irreducible representation of $SL(2, \mathbb{R})$ generated by self-adjoint extensions of operators (47), (48) and (49) specialized to our case. In fact it is possible to show that the space is reducible and is decomposable as $D^{1/4} \oplus D^{3/4}$. $D^{1/4}$ and $D^{3/4}$ are irreducible representations of the universal covering of $SL(2, \mathbb{R})$ (more precisely, they are unitary irreducible representation of a subgroup $Mp(2)$ of that universal covering called the *metaplectic group*). All that shows that very simple physical systems as a classical free particle or a harmonic oscillator are not so simple from the point of view of the conformal symmetry. The apparent intriguing result that the ground state of an harmonic oscillator can be seen as

⁵Concerning [13], Professor Kumar S.Gupta kindly pointed out to the authors that the pair of works [13] made use of the ground eigenvalue only in actual calculations and, the non existence of the other eigenvalues could in fact make the results found in the second paper stronger.

a thermal state of the associated free classical particle requires further analysis because of the complex action of the representation.

Coming back to the main stream of the work, we have shown that a free massive spinless particle in Rindler spacetime can be considered as an elementary $SL(2, \mathbb{R})$ invariant system. Such a result is a direct consequence of the spectral decomposition of the Hilbert space with respect to the Hamiltonian operator. The result is preserved if one changes the background far from the horizon, provided the spectrum of H and its degeneracy are not affected from those changes. We have also found that the simplest $SL(2, \mathbb{R})$ representation, that faithful, involves the presence of selected thermal states. However, the interplay between $SL(2, \mathbb{R})$ symmetry and the appearance of thermal states deserves further investigation. In particular, it is not clear if, inside the model, there is some direct constraint which fixes the adimensional parameter λ to determine the Hawking-Unruh-Fulling temperature. In fact, that distinguished value is imposed by the geometric background at quantum *field* theory level (Bisognano-Wichmann-Sewell's theorems).

We have also analyzed the case of AdS_2 background. In this case the local $SL(2, \mathbb{R})$ symmetry in the energy spectrum is still present. Once again, a particle can be seen as an elementary conformal invariant system and the local Killing time evolution is embodied in the $SL(2, \mathbb{R})$ symmetry. However, we have also shown that the $SL(2, \mathbb{R})$ symmetric background geometry has a nice interplay with the $SL(2, \mathbb{R})$ energy symmetry. Indeed, the background conformal representation pick out, and in fact is equivalent to, one of the possible irreducible energy $SL(2, \mathbb{R})$ representations. The choice depend on the value of the mass of the particle. In any case, the unique faithful $SL(2, \mathbb{R})$ representation is forbidden and no selected thermal states arise in this framework. This fact is in agreement with known results on quantum field states in nonbifurcate black hole background. In the AdS_2 background, the operator K_λ , which is responsible for the appearance of thermal states in the Rindler background, acquires a dynamical meaning. We have shown in details that, as earlier suggested in other works, it defines the Hamiltonian evoluter with respect to a appropriate global Killing time of the spacetime, provided a suiteable choice of the parameter λ is made.

Obviously the main issue which merits to be investigated concerns possible generalizations of these results to spacetime with dimension $d > 2$. Generalizations might involve the interplay between the angular degrees of freedom around a black hole and the energy spectrum of the particles. We expect that in some cases, the Hilbert space of a particle turns out to be a direct decomposition of (generally hidden) $SL(2, \mathbb{R})$ irreducible representations labeled by some discrete parameter related to the quantized angular momentum.

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